Discrete-Time Linear Periodic Realization in the Frequency Domain*

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ABSTRACT

Given an appropriate collection of periodic rational matrices, we characterize when it has a periodic realization. The state-space and the input-output invariant formulation of discrete-time periodic linear systems are involved.

1. INTRODUCTION AND NOTATION

The theory of realizations of invariant linear systems has been characterized in different ways. Thus, in the input-output model, it is well known that an input-output invariant application has an invariant realization if and only if its Markov parameters satisfy a recurrence equation (see Delchamps [2]). A

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different characterization can be given by means of the Hankel matrix associated with the Markov sequence, i.e., an input-output invariant application is realizable if and only if the corresponding infinite Hankel matrix has finite rank (see Sontag [9]). In addition, it is well known (see Kailath [5]) that the rank of the Hankel matrix equals the dimension of the minimal realization. Further, in the frequency domain, the existence of an invariant realization is equivalent to the existence and proper rationality of its transfer-matrix function (see Brocket [1], Rosenbrock [6]).

The extension of the above results for discrete-time periodic linear systems has recently been studied. Sánchez, Hernández, and Bru [8] introduce the invariant formulation of input-output periodic applications and give a necessary and sufficient condition for the existence of a periodic realization in terms of recurrence equations of its periodic Markov parameters. The same authors extend the characterization of the dimension of the minimal realization to the periodic case, defining the concept of periodic Hankel matrix (see [7]). Grasselli and Longhi [4] introduced the concept of the periodic transfer-matrix function for discrete-time periodic linear systems, by using the transfer matrices of the associated invariant systems, and they study the zeros and poles of that periodic matrix.

The main aim of this paper is to characterize when a set of rational periodic matrices has a periodic realization. We obtain a natural generalization of the above-mentioned result on invariant realizations in the frequency domain. More concretely, a collection of periodic rational matrices $\{H_s(z), s \in \mathbb{Z}\}$, $H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN \times mN}(z)$ is N-periodically realizable if and only if $H_{s+1}(z) = S(z)H_s(z)T(z)$, $s \in \mathbb{Z}$, where $S(z)$ and $T(z)$ are given below and $H_0(z)$ is proper with strictly lower block-triangular polynomial part with respect to the antidiagonal. In the proof of that characterization we use both invariant formulations of discrete-time linear periodic systems: the input-output invariant formulation and the state-space one.

The paper is structured as follows. First, we give some notation which we shall be using in the sequel. Section 2 contains a short description of the invariant formulation of a discrete linear N-periodic system (see [3, 8]) and some basic results of periodic realizations in the input-output model (see [8]). We end this section by defining the concept of a periodic collection of rational matrices periodically realizable. In Section 3, we give the relationship among the Laurent coefficients of the periodic collection of rational matrices and construct N invariant realizations of the finite family $H_0(z), H_1(z), \dots, H_{N-1}(z)$ from the Laurent coefficients of $H_0(z)$. Then, from the corresponding N invariant input-output applications we define an input-output periodic application satisfying the conditions for the existence of a periodic realization. We prove that this periodic realization realizes the initial collection of periodic rational matrices.

We will use the following notation. Given an arbitrary polynomial matrix $M(z) \in \mathbb{R}^{pN \times mN}[z]$, $M(z) = [m_{ij}(z)]$, we write $M(z)$ by blocks:

$$M(z) = ([M(z)]_{h,l}), \qquad h, l = 1, 2, \dots, N,$$

where

$$[M(z)]_{h,l} = \begin{bmatrix} m_{(h-1)p+1,(l-1)m+1}(z) & \cdots & m_{(h-1)p+1,lm}(z) \\ \vdots & & \vdots \\ m_{hp,(l-1)m+1}(z) & \cdots & m_{hp,lm}(z) \end{bmatrix}$$

$$\in \mathbb{R}^{p \times m}[z]. \tag{1.1}$$

2. PRELIMINARIES

Consider the discrete-time linear periodic system given by

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \\ y(k) &= C(k)u(k), \end{aligned} \tag{2.1}$$

where $A(k+N) = A(k) \in \mathbb{R}^{n \times n}$, $B(k+N) = B(k) \in \mathbb{R}^{n \times m}$, $C(k+N) = C(k) \in \mathbb{R}^{p \times n}$, $k \in \mathbb{Z}$, $N \in \mathbb{Z}^+$. Let us denote this system by $(C(\cdot), A(\cdot), B(\cdot))_N$.

Suppose that the system (2.1) starts at the zero initial state at time s, $s \in \mathbb{Z}$. Then the input-output application of the system (2.1), at time s, is given by

$$y(k+s) = \sum_{j=0}^{k-1} W_s(k, k-j)u(j+s), \qquad k \geqslant 1, \tag{2.2}$$

where $W_s(k,j) = C(k+s)\phi_A(k+s, k+s-j+1)B(k+s-j) \in \mathbb{R}^{p \times m}$, $j = 1, \dots, k$, are the Markov parameters of the periodic system (2.1), and

$\phi_A(\cdot, \cdot)$ is the state transition matrix

$$\phi_A(k, k_0) = A(k-1)A(k-2)\cdots A(k_0), \quad k > k_0,$$

$$\phi_A(k, k) = I_n, \quad k, k_0 \in \mathbb{Z}.$$

The Markov parameters $W_s(k, j)$ satisfy the following properties (see [7]):

(i) $W_{s+1}(k, j) = W_s(k+1, j)$ (*translation property*),
(ii) $W_s(k+N, j) = W_s(k, j)$ (*periodicity property*).

We will refer to these properties as conditions (i) and (ii).

On the other hand, it is well known [3] that for any $s \in \mathbb{Z}$, there exists an invariant system associated with the periodic system (2.1),

$$x_s(k+1) = A_s x_s(k) + B_s u_s(k),$$

$$y_s(k) = C_s x_s(k) + D_s u_s(k), \tag{2.3}$$

where

$$x_s(k) = x(s+kN),$$

$$u_s(k) = \operatorname{col}[u(s+kN+N-1), u(s+kN+N-2), \ldots, u(s+kN)],$$

$$y_s(k) = \operatorname{col}[y(s+kN), y(s+kN+1), \ldots, y(s+kN+N-1)],$$

and where $A_s \in \mathbb{R}^{n\times n}$, $B_s \in \mathbb{R}^{n\times mN}$, $C_s \in \mathbb{R}^{pN\times n}$ are given by

$$A_s = \phi_A(s+N, s),$$

$$B_s = [B(s+N-1), \phi_A(s+N, s+N-1)B(s+N-2), \ldots,$$
$$\phi_A(s+N, s+1)B(s)],$$

$$C_s = \operatorname{col}[C(s), C(s+1)\phi_A(s+1, s), \ldots,$$
$$C(s+N-1)\phi_A(s+N-1, s)].$$

The matrix $D_s \in \mathbb{R}^{pN \times mN}$, given by the following expressions, is strictly lower block-triangular with respect to the antidiagonal:

$$D_s = \left[D_{ij}^s \right], \quad D_{ij}^s \in \mathbb{R}^{p \times m}, \qquad i, j = 1, \dots, N,$$

$$D_{ij}^s = \begin{cases} O, & i + j < N + 2, \\ C(s + i - 1)\phi_A(s + i - 1, N + s - j + 1)B(N + s - j), \\ \qquad i + j \geqslant N + 2. \end{cases}$$

We denote this invariant system by (C_s, A_s, B_s, D_s). From the periodicity of the system (2.1), note that $(C_{s+N}, A_{s+N}, B_{s+N}, D_{s+N}) = (C_s, A_s, B_s, D_s)$, $s \in \mathbb{Z}$. Then we only have N different associated invariant linear systems, (C_s, A_s, B_s, D_s), $s = 0, 1, \dots, N - 1$. These systems are called the invariant formulation of the periodic system (2.1). The corresponding input-output applications are given by

$$y_s(k) = \sum_{j=0}^{k} W_s(k - j)u_s(j), \qquad k \geqslant 0, \quad s = 0, 1, \dots, N - 1, \quad (2.4)$$

where $\{W_s(k)\}_{k \geqslant 0} \subset \mathbb{R}^{pN \times mN}$, $W_s(k) = C_s A_s^{k-1} B_s$, $k \geqslant 1$, $W_s(0) = D_s$ are the Markov parameters of the system (2.3).

Now we recall some technical results on input-output periodic applications given in [8], which are the starting points of our results.

DEFINITION 2.1.

(a) An input-output application

$$y(k + s) = \sum_{j=0}^{k-1} U_s(k, k - j)u(j + s), \qquad k \geqslant 1, \quad s \in \mathbb{Z}, \quad (2.5)$$

is *N-periodic* if the corresponding Markov sequences $\{U_s(k, j) : k \geqslant 1,\ j = 1, \dots, k\} \subset \mathbb{R}^{p \times m}$, $s \in \mathbb{Z}$ satisfy conditions (i) and (ii).

(b) A periodic system $(G(\cdot), E(\cdot), F(\cdot))_N$ is said to be a *periodic realization* of the input-output periodic application (2.5) if for each $s \in \mathbb{Z}$ one has $U_s(k, j) = G(k + s)\phi_E(k + s, k + s - j + 1)F(k + s - j)$, $k \geqslant 1$, $j = 1, \dots, k$. The dimension of the matrix $E(\cdot)$ is called the dimension of the realization.

Now we consider the invariant formulation of an input-output periodic application.

DEFINITION 2.2. Consider the input-output N-periodic application (2.5) defined by the Markov sequences $\{U_s(k,j): k \geqslant 1,\ j = 1,\dots,k\} \subset \mathbb{R}^{p\times m}$, $s \in \mathbb{Z}$. For each $s \in \mathbb{Z}$ the *associated input-output invariant application* is defined by a Markov sequence $\{U_s(k): k \geqslant 0\} \subset \mathbb{R}^{pN\times mN}$, where the matrix $U_s(k)$ is defined by blocks as follows:

$$U_s(k) = \left([U_s(k)]_{\alpha,\beta}\right), \qquad \alpha,\beta = 1,2,\dots,N, \tag{2.6}$$

with

$$[U_s(k)]_{\alpha,\beta} = U_s(kN + \alpha - 1, (k-1)N + \alpha + \beta - 1). \tag{2.7}$$

We assume that $U_s(k,j) = O$ if $k \leqslant 0$ or $j \leqslant 0$.

From the translation and periodicity properties of $U_s(k,j)$ it is easy to prove that $U_{s+N}(k) = U_s(k)$, $s \in \mathbb{Z}$. Then we only have N different associated input-output invariant applications $\{U_s(k): k \geqslant 0\}$, $s = 0,1,\dots,N-1$. These applications are called the *invariant formulation* of the periodic applications (2.5).

The following result shows that both input-output and state-space invariant formulations are consistent. The proof is straightforward; see [8].

PROPOSITION 2.1. *Consider the input-output N-periodic application* (2.5) *defined by the Markov sequences $\{U_s(k,j): k \geqslant 1,\ j = 1,\dots,k\} \subset \mathbb{R}^{p\times m}$, $s \in \mathbb{Z}$, satisfying properties* (i) *and* (ii). *If $(G(\cdot), E(\cdot), F(\cdot))_N$ is a periodic realization of* (2.5), *then the associated invariant system (G_s, E_s, F_s, H_s) is a realization of the input-output invariant application defined by the Markov parameters* (2.6), (2.7).

A characterization of the existence of periodic realizations is given in the following theorem.

THEOREM 2.1. *A necessary and sufficient condition for the existence of a periodic realization of the input-output periodic application* (2.5) *is that there exists $r \in \mathbb{Z}$ and there are scalars $a_0, a_1, \dots, a_{r-1} \in \mathbb{R}$ such that*

$$\begin{aligned} U_s(k + rN, j + rN) &+ a_{r-1}U_s(k + (r-1)N, j + (r-1)N) \\ &+ \cdots + a_0 U_s(k,j) = 0 \end{aligned} \tag{2.8}$$

for each $s \in \mathbb{Z}$ *and for every* $k \geqslant 1$, *where* $j = 1, \dots, k$ *if* $1 \leqslant k < N$, *and* $j = k - N + 1, \dots, k$ *if* $k \geqslant N$.

Proof. *Necessary condition*: If $(G(\cdot), E(\cdot), F(\cdot))_N$ is a periodic realization of the input-output periodic application (2.5), then for each $s \in \mathbb{Z}$, $U_s(k, j) = G(k+s)\phi_E(k+s, k+s-j+1)F(k+s-j)$, $k \geqslant 1$, $j = 1, \dots, k$. By Proposition 2.1, for each $s \in \mathbb{Z}$, the invariant system (G_s, E_s, F_s, H_s) is an invariant realization of the corresponding input-output invariant application, satisfying $U_s(p) = G_s E_s^{p-1} F_s$, $p \geqslant 1$, $U_s(0) = H_s$, where the Markov sequence $\{U_s(p),\ p \geqslant 0\} \subset \mathbb{R}^{pN \times mN}$ is constructed by Definition 2.2.

From the well-known characterization for invariant systems (see Sontag [9]), there exist $r \in \mathbb{Z}^+$ and scalars $a_0, a_1, \dots, a_{r-1} \in \mathbb{R}$ such that

$$U_s(p+r) + a_{r-1}U_s(p+r-1) + \cdots + a_0 U_s(p) = O, \quad p \geqslant 1, \quad s \in \mathbb{Z},$$

where $a_0, a_1, \dots, a_{r-1}$ are the coefficients of the characteristic polynomial of the matrix E_s.

If we consider the partition of $U_s(p) = [U_s(p)]_{\alpha,\beta}$ into blocks of size $p \times m$, these blocks also satisfy the recurrence equation

$$[U_s(p+r)]_{\alpha,\beta} + a_{r-1}[U_s(p+r-1)]_{\alpha,\beta} + \cdots + a_0[U_s(p)]_{\alpha,\beta} = O,$$
$$\alpha, \beta = 1, \dots, N, \quad p \geqslant 1, \quad s \in \mathbb{Z}.$$

From Definition 2.2 we obtain

$$U_s(k+rN, j+rN) + a_{r-1}U_s(k+(r-1)N, j+(r-1)N) + \cdots + a_0 U_s(k,j) = O, \tag{2.9}$$

where $k = p + \alpha - 1$, $j = k - N + \beta$.

In order to extend the expression (2.9) to $1 \leqslant k < N$ and $j = 1, \dots, k$, it is sufficient to consider the periodicity and translation properties, (i) and (ii), of the matrices $\{U_s(k,j) : k \geqslant 1,\ j = 1, \dots, k\} \subset \mathbb{R}^{p \times m}$, $s \in \mathbb{Z}$. Consider (2.9) at time $s + l$, $1 \leqslant l < N$, and $k = N$:

$$U_{s+l}(N+rN, j+rN) + a_{r-1}U_{s+l}(N+(r-1)N, j+(r-1)N) + \cdots + a_0 U_{s+l}(N, j) = O,$$
$$j = 1, \dots, N, \quad k = N, \quad s \in \mathbb{Z}.$$

From (i) and (ii), we conclude

$$U_s(l + rN, j + rN) + a_{r-1}U_s(l + (r-1)N, j + (r-1)N)$$
$$+ \cdots + a_0 U_s(l, j) = O,$$
$$j = 1, \ldots, l, \quad 1 \leqslant l < N, \quad s \in \mathbb{Z}.$$

Sufficient condition: Consider $\{U_s(k, j) : k \geqslant 1,\ j = 1, \ldots, k\} \subset \mathbb{R}^{p \times m}$, $s \in \mathbb{Z}$, satisfying the recurrence equation (2.8). We define the periodic system $(C_0(\cdot), A_0(\cdot), B_0(\cdot))_N$ where for every $k \in \mathbb{Z}$, $A_0(k)$ is the lower block-companion matrix

$$A_0(k) = \begin{bmatrix} O & I_p & O & \cdots & O \\ O & O & I_p & \cdots & O \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & O & \cdots & I_p \\ * & * & * & \cdots & * \end{bmatrix} \in \mathbb{R}^{rpN \times rpN},$$

with the last row given by

$$\left(-a_0 I_p, O, \ldots, O, -a_1 I_p, O, \ldots, O, \ldots, -a_{r-1} I_p, O, \ldots, O\right),$$

and

$$C_0(k) = \left[I_p, O, \ldots, O\right] \in \mathbb{R}^{p \times rpN},$$
$$B_0(k) = \operatorname{col}\left[U_k(1,1), U_k(2,2), \ldots, U_k(rN, rN)\right] \in \mathbb{R}^{rpN \times m}. \quad (2.10)$$

By construction, the matrices $A_0(k)$ and $C_0(k)$ are time-invariant. We denote them by $A_0 = A_0(k)$ and $C_0 = C_0(k)$. From the periodicity property (ii) of $U_s(k, j)$ we obtain that the matrix $B_0(k)$ is periodic: $B_0(k + N) = B_0(k)$ for all $k \in \mathbb{Z}$.

The characteristic polynomial of matrix A_0 is given by

$$p(\lambda) = |\lambda I - A_0| = \pm\left(a_0 + a_1\lambda^N + \cdots + a_{r-1}\lambda^{(r-1)N} + \lambda^{rN}\right)^p,$$

and the minimal polynomial by

$$m(\lambda) = a_0 + a_1\lambda^N + \cdots + a_{r-1}\lambda^{(r-1)N} + \lambda^{rN}.$$

As A_0 satisfies its minimal equation, we obtain

$$a_0 I + a_1 A_0^N + \cdots + a_{r-1} A_0^{(r-1)N} + A_0^{rN} = O.$$

Note that

$$\begin{aligned} C_0 &= \left[I_p, O, \dots, O \right], \\ C_0 A_0 &= \left[O, I_p, \dots, O \right], \\ &\vdots \\ C_0 A_0^{rN-1} &= \left[O, O, \dots, I_p \right], \\ C_0 A_0^{rN} &= -a_0 C_0 - a_1 C_0 A_0^N - \cdots - a_{r-1} C_0 A_0^{(r-1)N}. \end{aligned} \tag{2.11}$$

Now, we prove that the periodic system $(C_0(\cdot), A_0(\cdot), B_0(\cdot))_N$ is a periodic realization of the input-output periodic application (2.5), i.e.

$$\begin{aligned} U_s(k,j) &= C_0(k+s)\phi_{A_0}(k+s, k+s-j+1) B_0(k+s-j) \\ &= C_0 A_0^{j-1} B_0(k+s-j), \qquad j = 1, \dots, k, \quad k \geqslant 1, \quad s \in \mathbb{Z}. \end{aligned} \tag{2.12}$$

We shall prove (2.12) by induction on the index j. From (2.11), the definition of $B_0(k)$ in (2.10), and translation property (i), we have

$j = 1$:

$$\begin{aligned} C_0 B_0(k+s-1) &= \left[I_p, O, \dots, O \right] B_0(k+s-1) \\ &= U_{k+s-1}(1,1) = U_s(k,1), \qquad k \geqslant 1, \end{aligned}$$

$j = 2$:

$$C_0 A_0 B_0(k+s-2) = [O, I_p, \dots, O] B_0(k+s-2)$$
$$= U_{k+s-2}(2,2) = U_s(k,2), \qquad k \geqslant 2,$$
$$\vdots$$

$j = rN$:

$$C_0 A_0^{rN-1} B_0(k+s-rN) = [O, O, \dots, I_p] B_0(k+s-rN)$$
$$= U_{k+s-rN}(rN, rN) = U_s(k, rN), \qquad k \geqslant rN.$$

Now, we suppose that (2.12) is satisfied for $j \geqslant rN : C_0 A_0^{j-1} B_0(k+s-j) = U_s(k,j)$, $k \geqslant j$. Then we prove that (2.12) is satisfied for $j+1$. If $j = rN + h$ with $h \geqslant 0$ then $j + 1 = rN + h + 1$. From the minimal equation of A_0 and the induction hypothesis, we obtain

$$C_0 A_0^j B_0(k+s-j-1) = C_0 A_0^{rN+h} B_0(k+s-rN-h-1)$$
$$= -a_0 C_0 A_0^h B_0(k+s-rN-h-1) - \cdots$$
$$- a_{r-1} C_0 A_0^{(r-1)N+h} B_0(k+s-rN-h-1)$$
$$= -a_0 U_s(k-rN, h+1)$$
$$- \cdots - a_{r-1} U_s(k-N, (r-1)N+h+1).$$

Applying the recurrence equation (2.8), we conclude

$$C_0 A_0^j B_0(k+s-j-1) = U_s(k, rN+h+1) = U_s(k, j+1),$$
$$k \geqslant j+1. \qquad \blacksquare$$

As a consequence of Theorem 2.1 we state the following result (see [8]).

PROPOSITION 2.2. *Consider* $\{U_s(k,j) : k \geqslant 1,\ j = 1, \dots, k\} \subset \mathbb{R}^{p \times m}$, $s \in \mathbb{Z}$, *satisfying properties* (i) *and* (ii). $U_s(k,j)$ *satisfies the recurrence equation in Theorem* 2.1 *if and only if* $U_s(k)$ *verifies the same recurrence equation, that is,*

$$U_s(k+r) + a_{r-1} U_s(k+r-1) + \cdots + a_0 U_s(k) = O, \qquad k \geqslant 1.$$

By means of the following definition we shall construct an input-output N-periodic application associated to N given input-output invariant applications.

DEFINITION 2.3. Let $\{V_s(k): k \geqslant 0\} \subset \mathbb{R}^{pN \times mN}$, $s = 0, 1, \dots, N-1$, be N Markov sequences defining N input-output invariant applications. Consider the N-periodic extension of these Markov sequences given by $V_{s+N}(k) = V_s(k)$, $s \in \mathbb{Z}$, and a partition of $V_s(k)$ into blocks of size $p \times m$:

$$V_s(k) = \left([V_s(k)]_{\alpha,\beta}\right), \qquad \alpha, \beta = 1, 2, \dots, N.$$

Let the matrix sequences

$$\{V_s(k,j) : k \geqslant 1, j = 1, \dots, k\} \subset \mathbb{R}^{p \times m}, \qquad s \in \mathbb{Z}, \tag{2.13}$$

be defined by means of the following expressions. Set $k = \gamma N + i - 1$, $\gamma \geqslant 0$, $i = 1, \dots, N$. If $1 \leqslant k \leqslant N$ ($\gamma = 0, i = 2, \dots, N$) then

$$V_s(k,j) = [V_s(0)]_{i, j+N-i+1}. \tag{2.14}$$

If $k \geqslant N$ ($\gamma \geqslant 1$) then

$$V_s(k,j) = \begin{cases} [V_s(0)]_{i,j+N-i+1}, & j = 1, \dots, i-1, \\ [V_s(h)]_{i,j-(h-1)N-i+1} & (h-1)N + i \leqslant j < hN + i - 1, \\ & \qquad 1 \leqslant h \leqslant \gamma. \end{cases} \tag{2.15}$$

The following result characterizes when the matrix sequences defined by (2.13)–(2.15) satisfy both properties (i) and (ii). So these Markov sequences will define an input-output *N-periodic* application. The proof is rather technical, and can be found in [8].

LEMMA 2.1. *Let* $\{V_s(k,j) : k \geqslant 1,\ j = 1,\dots,k\} \subset \mathbb{R}^{p\times m}$, $s \in \mathbb{Z}$, *be the Markov sequences introduced in Definition* 2.3. *If* $\{V_s(k) : k \geqslant 0\}$ *satisfies*

$$[V_{s+1}(k)]_{\alpha,\beta} = [V_s(k)]_{\alpha+1,\beta-1}, \qquad 1 \leqslant \alpha \leqslant N-1, \quad 2 \leqslant \beta \leqslant N, \tag{2.16a}$$

$$[V_{s+1}(k)]_{\alpha,1} = [V_s(k-1)]_{\alpha+1,N}, \qquad 1 \leqslant \alpha \leqslant N-1, \tag{2.16b}$$

$$[V_{s+1}(k)]_{N,\beta} = [V_s(k+1)]_{1,\beta-1}, \qquad 2 \leqslant \beta \leqslant N, \tag{2.16c}$$

$$[V_{s+1}(k)]_{N,1} = [V_s(k)]_{1,N}, \tag{2.16d}$$

then $V_{s+1}(k,j) = V_s(k+1,j)$ *and* $V_s(k+N,j) = V_s(k,j)$.

COROLLARY 2.1 (see [8]). *Let* $\{V_s(k) : k \geqslant 0\} \subset \mathbb{R}^{pN\times mN}$, $s = 0,1,\dots,N-1$, *be* N *Markov sequences defining* N *input-output invariant applications such that satisfy the condition* (2.16) *If* $V_s(0) = ([V_s(0)]_{\alpha,\beta})$ *satisfies* $[V_s(0)]_{\alpha,\beta} = O$, $\alpha + \beta < N + 2$ ($V_s(0)$ *is strictly lower block-triangular with respect to the antidiagonal*), *then the invariant formulation of the input-output periodic application introduced in Definition* 2.3 *is given by* $\{V_s(k) : k \geqslant 0\}$, $s = 0,1,\dots,N-1$.

Next, consider the transfer matrix of the invariant system (2.3) associated to the periodic system (2.1), at time s:

$$G_s(z) = C_s(zI - A_s)^{-1}B_s + D_s. \tag{2.17}$$

Note that $G_s(z)$ is a proper rational matrix such that its polynomial part, D_s, is strictly lower block-triangular with respect to the antidiagonal.

From the periodicity of the system (2.1), the collection of transfer matrices (2.17) satisfies $G_{s+N}(z) = G_s(z)$. Grasselli and Longhi [4] showed that these transfer matrices satisfy the following relation:

$$G_{s+1}(z) = S(z)G_s(z)T(z), \qquad s \in \mathbb{Z},\ z \neq 0, \tag{2.18}$$

where

$$S(z) = \begin{bmatrix} O & I_{(N-1)p} \\ zI_p & O \end{bmatrix}, \qquad T(z) = \begin{bmatrix} O & I_{(N-1)m} \\ z^{-1}I_m & O \end{bmatrix}. \tag{2.19}$$

As we mentioned in the introduction, our goal is to study the following problem: Given a collection of periodic rational matrices $\{H_s(z),\ s \in \mathbb{Z}\}$, $H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN \times mN}(z)$, when does an N-periodic realization for these rational matrices exist? We study this question in the next section. First, we introduce the definition of a periodic realization of a given periodic collection of rational matrices.

DEFINITION 2.4. An N-periodic system $(G(\cdot), E(\cdot), F(\cdot))_N$ is called a periodic realization of the periodic collection of rational matrices given by $\{H_s(z),\ s \in \mathbb{Z}\}$, $H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN \times mN}(z)$, if $H_s(z) = G_s(zI - E_s)^{-1}F_s + H_s$, $s \in \mathbb{Z}$, where (G_s, E_s, F_s, H_s) is the invariant system associated with the given periodic system at time s. The dimension of $E(\cdot)$ is called the dimension of the realization.

3. PERIODIC REALIZATION OF A COLLECTION OF PERIODIC RATIONAL MATRICES

3.1. *Collection of Invariant Realizations*

Consider a sequence of periodic rational matrices

$$\{H_s(z),\ s \in \mathbb{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN \times mN}(z), \quad N \in \mathbb{Z}^+, \tag{3.1.1}$$

satisfying

$$H_{s+1}(z) = S(z) H_s(z) T(z), \tag{3.1.2}$$

where $S(z)$ and $T(z)$ are given in (2.19). We assume that $H_0(z)$ is a proper rational matrix with strictly lower block-triangular polynomial part with respect to the antidiagonal. Then $H_0(z) = \tilde{H}_0(z) + \tilde{D}_0$, where $\tilde{H}_0(z)$ is a strictly proper rational matrix and

$$\left[\tilde{D}_0\right]_{\alpha,\beta} = O \qquad \text{for} \quad \alpha + \beta < N + 2. \tag{3.1.3}$$

Let $d_0(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_0$ be the monic least common denominator of all the elements of $H_0(z)$. Then $H_0(z)$ can be expressed as

$$H_0(z) = \frac{N_0(z)}{d_0(z)}, \qquad \delta N_0(z) \leqslant r, \tag{3.1.4}$$

where $\delta N_0(z)$ denotes the degree of the matrix polynomial $N_0(z)$. From (3.1.3), $N_0(z)$ is such that

$$\delta[N_0(z)]_{\alpha,\beta} < r \qquad \text{for} \quad \alpha + \beta < N + 2. \tag{3.1.5}$$

Then we give the following result.

PROPOSITION 3.1. *Suppose that the collection of periodic rational matrices* (3.1.1) *satisfies* (3.1.2). *If $H_0(z)$ is proper with strictly lower block-triangular polynomial part with respect to the antidiagonal, then $H_s(z)$ is also proper with polynomial part of the same type as $H_0(z)$ and $H_s(z) = N_s(z)/d_s(z)$, where*

See page 315 *for equation* (3.1.6).

and $d_s(z) = zd_0(z)$, $s = 1, 2, \dots, N-1$.

Proof. From (3.1.2) and (3.1.4), we have

$$\begin{aligned}
d_0(z)H_1(z) &= S(z)N_0(z)T(z),\\
d_0(z)H_2(z) &= d_0(z)S(z)H_1(z)T(z) = d_0(z)S^2(z)N_0(z)T^2(z),\\
&\vdots\\
d_0(z)H_{N-1}(z) &= d_0(z)S(z)H_{N-2}(z)T(z)\\
&= \cdots = d_0(z)S^{N-1}(z)N_0(z)T^{N-1}(z).
\end{aligned}$$

In general

$$d_0(z)H_s(z) = S^s(z)N_0(z)T^s(z), \qquad s = 1, 2, \dots, N-1.$$

The premultiplication of the matrix $N_0(z)$ by $S(z)$ is equivalent to the following operations on the rows of $N_0(z)$: the ith row, $i = 2, \dots, N$, takes the place of the $(i-1)$th row, and the 1st row is multiplied by z and takes the place of the Nth row. The postmultiplication of the matrix $N_0(z)$ by $T(z)$ is equivalent to the following operations on the columns of $N_0(z)$: the jth column, $j = 1, \dots, N-1$, takes the place of the $(j+1)$th column, and the Nth column is multiltiplied by z^{-1} and takes the place of the 1st column.

$$N_s(z) = \left[\begin{array}{ccc|ccc} [N_0(z)]_{s+1,N-s+1} & \cdots & [N_0(z)]_{s+1,N} & z[N_0(z)]_{s+1,1} & \cdots & z[N_0(z)]_{s+1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ [N_0(z)]_{N,N-s+1} & \cdots & [N_0(z)]_{N,N} & z[N_0(z)]_{N,1} & \cdots & z[N_0(z)]_{N,N-s} \\ \hline z[N_0(z)]_{1,N-s+1} & \cdots & z[N_0(z)]_{1,N} & z^2[N_0(z)]_{1,1} & \cdots & z^2[N_0(z)]_{1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ z[N_0(z)]_{s,N-s+1} & \cdots & z[N_0(z)]_{s,N} & z^2[N_0(z)]_{s,1} & \cdots & z^2[N_0(z)]_{s,N-s} \end{array}\right], \tag{3.1.6}$$

Then we have

$$d_0(z)H_s(z) = \left[\begin{array}{ccc|ccc} z^{-1}[N_0(z)]_{s+1,N-s+1} & \cdots & z^{-1}[N_0(z)]_{s+1,N} & [N_0(z)]_{s+1,1} & \cdots & [N_0(z)]_{s+1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ z^{-1}[N_0(z)]_{N,N-s+1} & \cdots & z^{-1}[N_0(z)]_{N,N} & [N_0(z)]_{N,1} & \cdots & [N_0(z)]_{N,N-s} \\ \hdashline [N_0(z)]_{1,N-s+1} & \cdots & [N_0(z)]_{1,N} & z[N_0(z)]_{1,1} & \cdots & z[N_0(z)]_{1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ [N_0(z)]_{s,N-s+1} & \cdots & [N_0(z)]_{s,N} & z[N_0(z)]_{s,1} & \cdots & z[N_0(z)]_{s,N-s} \end{array}\right],$$

$$s = 1, 2, \ldots, N-1.$$

By multiplying the above expression by z, we obtain that $H_s(z)$ can be written as $H_s(z) = N_s(z)/d_s(z)$, where $N_s(z)$ is the polynomial matrix given in (3.1.6), and $d_s(z) = zd_0(z)$. The degree of $d_s(z)$ is equal to $r + 1$ and $\delta N_s(z) \leqslant r + 1$. From (3.1.4), one has $\delta[N_0(z)]_{\alpha,\beta} \leqslant r$, and by (3.1.5) one deduces

$$\delta \begin{bmatrix} [N_0(z)]_{1,1} & \cdots & [N_0(z)]_{1,N-s} \\ \vdots & & \vdots \\ [N_0(z)]_{s,1} & \cdots & [N_0(z)]_{s,N-s} \end{bmatrix} < r.$$

Hence $H_s(z)$ is a proper rational matrix.

From (3.1.5) and (3.1.6) we deduce that the antidiagonal blocks and the upper antidiagonal blocks of $N_s(z)$ have degree less than $r + 1$. Then $H_s(z) = \tilde{H}_s(z) + \tilde{D}_s$, where $\tilde{H}_s(z)$ is a strictly proper rational matrix and $[\tilde{D}_s]_{\alpha,\beta} = O$ for $\alpha + \beta < N + 2$. ■

In a similar way, the Laurent series expansion of $H_s(z)$ follows from the Laurent series expansion of $H_0(z)$,

$$H_0(z) = \sum_{n=0}^{\infty} L_n^0 z^{-n} = \sum_{n=1}^{\infty} L_n^0 z^{-n} + L_0^0 = \tilde{H}_0(z) + \tilde{D}_0. \quad (3.1.7)$$

Note that

$$[L_0^0]_{\alpha,\beta} = [\tilde{D}_0]_{\alpha,\beta} = O \qquad \text{for} \quad \alpha + \beta < N + 2. \quad (3.1.8)$$

From (3.1.2) we have that

$$H_1(z) = \sum_{n=0}^{\infty} S(z) L_n^0 T(z) z^{-n}, \quad (3.1.9)$$

where

$$S(z)L_n^0T(z) = \left[\begin{array}{c|ccc} z^{-1}[L_n^0]_{2,N} & [L_n^0]_{2,1} & \cdots & [L_n^0]_{2,N-1} \\ \vdots & \vdots & & \vdots \\ z^{-1}[L_n^0]_{N,N} & [L_n^0]_{N,1} & \cdots & [L_n^0]_{N,N-1} \\ \hline [L_n^0]_{1,N} & z[L_n^0]_{1,1} & \cdots & z[L_n^0]_{1,N-1} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} O & O & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & O \\ \hline O & \left[L_n^0\right]_{1,1} & \cdots & \left[L_n^0\right]_{1,N-1} \end{array}\right] z$$

$$+ \left[\begin{array}{c|ccc} O & \left[L_n^0\right]_{2,1} & \cdots & \left[L_n^0\right]_{2,N-1} \\ \vdots & \vdots & & \vdots \\ O & \left[L_n^0\right]_{N,1} & \cdots & \left[L_n^0\right]_{N,N-1} \\ \hline \left[L_n^0\right]_{1,N} & O & \cdots & O \end{array}\right]$$

$$+ \left[\begin{array}{c|ccc} \left[L_n^0\right]_{2,N} & O & \cdots & O \\ \vdots & \vdots & & \vdots \\ \left[L_n^0\right]_{N,N} & O & \cdots & O \\ \hline O & O & \cdots & O \end{array}\right] z^{-1}.$$

Using this expression in the expansion (3.1.9), we obtain

$$H_1(z) = \sum_{n=0}^{\infty} L_n^1 z^{-n} = \sum_{n=1}^{\infty} L_n^1 z^{-n} + L_0^1 = \tilde{H}_1(z) + \tilde{D}_1,$$

where

$$L_n^1 = \left[\begin{array}{c|ccc} \left[L_{n-1}^0\right]_{2,N} & \left[L_n^0\right]_{2,1} & \cdots & \left[L_n^0\right]_{2,N-1} \\ \vdots & \vdots & & \vdots \\ \left[L_{n-1}^0\right]_{N,N} & \left[L_n^0\right]_{N,1} & \cdots & \left[L_n^0\right]_{N,N-1} \\ \hline \left[L_n^0\right]_{1,N} & \left[L_{n+1}^0\right]_{1,1} & \cdots & \left[L_{n+1}^0\right]_{1,N-1} \end{array}\right], \qquad n \geqslant 1,$$

and, from (3.1.8)

$$L_0^1 = \tilde{D}_1 = \left[\begin{array}{c|cccc} O & O & O & \cdots & O \\ O & O & O & \cdots & [L_0^0]_{3,N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & [L_0^0]_{N,2} & \cdots & [L_0^0]_{N,N-1} \\ \hline O & [L_1^0]_{1,1} & [L_1^0]_{1,2} & \cdots & [L_1^0]_{1,N-1} \end{array}\right].$$

Using the above results, it is easy to prove, by recurrence, the next lemma, which is of central importance in the following.

LEMMA 3.1.1. *Consider the collection of periodic rational matrices* (3.1.1) *satisfying* (3.1.2), *and suppose that* $H_0(z)$ *is proper with strictly lower block-triangular polynomial part with respect to the antidiagonal. If the Laurent expansion of* $H_0(z)$ *is given by* (3.1.7), *then*

$$H_s(z) = \sum_{n=0}^{\infty} L_n^s z^{-n} = \sum_{n=1}^{\infty} L_n^s z^{-n} + L_0^s = \tilde{H}_s(z) + \tilde{D}_s,$$

$$s = 1, 2, \ldots, N-1, \quad (3.1.10)$$

where

$$L_n^s = \left[\begin{array}{ccc|ccc} [L_{n-1}^0]_{s+1,N-s+1} & \cdots & [L_{n-1}^0]_{s+1,N} & [L_n^0]_{s+1,1} & \cdots & [L_n^0]_{s+1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ [L_{n-1}^0]_{N,N-s+1} & \cdots & [L_{n-1}^0]_{N,N} & [L_n^0]_{N,1} & \cdots & [L_n^0]_{N,N-s} \\ \hline [L_n^0]_{1,N-s+1} & \cdots & [L_n^0]_{1,N} & [L_{n+1}^0]_{1,1} & \cdots & [L_{n+1}^0]_{1,N-s} \\ \vdots & & \vdots & \vdots & & \vdots \\ [L_n^0]_{s,N-s+1} & \cdots & [L_n^0]_{s,N} & [L_{n+1}^0]_{s,1} & \cdots & [L_{n+1}^0]_{s,N-s} \end{array}\right],$$

$$n \geqslant 1, \quad (3.1.11)$$

and

$$L_0^s = \tilde{D}_s$$

$$= \left[\begin{array}{cccc|cccc} O & O & \cdots & O & O & O & \cdots & O \\ O & O & \cdots & O & O & O & \cdots & [L_0^0]_{s+2,N-s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & O & O & [L_0^0]_{N,2} & \cdots & [L_0^0]_{N,N-s} \\ \hline O & O & \cdots & O & [L_1^0]_{1,1} & [L_1^0]_{1,2} & \cdots & [L_1^0]_{1,N-s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & [L_0^0]_{s,N-s+2} & \cdots & [L_0^0]_{s,N} & [L_1^0]_{s,1} & [L_1^0]_{s,2} & \cdots & [L_1^0]_{s,N-s} \end{array}\right]. \tag{3.1.12}$$

Since $H_0(z) \in \mathbb{R}^{pN \times mN}(z)$ is proper, $H_0(z) = N_0(z)/d_0(z)$, $d_0(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_0$, there exists an observable canonical invariant realization (see [1]), $H_0(z) = \tilde{C}_0(zI - \tilde{A}_0)^{-1}\tilde{B}_0 + \tilde{D}_0$, denoted by $(\tilde{C}_0, \tilde{A}_0, \tilde{B}_0, \tilde{D}_0)$, where

$$\tilde{A}_0 = \begin{bmatrix} O_{pN} & I_{pN} & O_{pN} & \cdots & O_{pN} \\ O_{pN} & O_{pN} & I_{pN} & \cdots & O_{pN} \\ \vdots & \vdots & \vdots & & \vdots \\ O_{pN} & O_{pN} & O_{pN} & \cdots & I_{pN} \\ -a_0 I_{pN} & -a_1 I_{pN} & -a_2 I_{pN} & \cdots & -a_{r-1} I_{pN} \end{bmatrix} \in \mathbb{R}^{rpN \times rpN}, \tag{3.1.13}$$

$$\tilde{B}_0 = \begin{bmatrix} L_1^0 \\ L_2^0 \\ \vdots \\ L_r^0 \end{bmatrix}, \qquad \tilde{C}_0 = \begin{bmatrix} I_{pN} & O_{pN} & \cdots & O_{pN} \end{bmatrix}, \qquad \tilde{D}_0 = L_0^0. \tag{3.1.14}$$

From this realization and Lemma 3.1.1 we obtain the following result.

PROPOSITION 3.1.2. *If the collection of periodic rational matrices* (3.1.1) *satisfies* (3.1.2) *and* $H_0(z)$ *is proper with strictly lower block-triangular polynomial part with respect to the antidiagonal, then for each* $s = 1, 2, \ldots, N-1$:

(i) *There exists an observable canonical invariant realization* $(\tilde{C}_s, \tilde{A}_s, \tilde{B}_s, \tilde{D}_s)$ *of* $H_s(z)$ *given by*

$$\tilde{A}_s = \begin{bmatrix} O_{pN} & I_{pN} & O_{pN} & \cdots & O_{pN} \\ O_{pN} & O_{pN} & I_{pN} & \cdots & O_{pN} \\ \vdots & \vdots & \vdots & & \vdots \\ O_{pN} & O_{pN} & O_{pN} & \cdots & I_{pN} \\ O_{pN} & -a_0 I_{pN} & -a_1 I_{pN} & \cdots & -a_{r-1} I_{pN} \end{bmatrix} \in \mathbb{R}^{(r+1)pN \times (r+1)pN}, \tag{3.1.15}$$

$$\tilde{B}_s = \begin{bmatrix} L_1^s \\ L_2^s \\ \vdots \\ L_r^s \\ L_{r+1}^s \end{bmatrix}, \qquad \tilde{C}_s = \begin{bmatrix} I_{pN} & O_{pN} & O_{pN} & \cdots & O_{pN} \end{bmatrix}, \tag{3.1.16}$$

and $\tilde{D}_s$ *is defined by the expression* (3.1.12).

(ii) *Further, the matrices* $\tilde{D}_s$ *and* $\tilde{D}_{s+1}$ *are related by the following expression*:

$$\tilde{D}_S \begin{bmatrix} O_{(N-1)m \times m} & I_{(N-1)m} \\ O_m & O_{m \times (N-1)m} \end{bmatrix} = \begin{bmatrix} O_{p \times (N-1)p} & O_p \\ I_{(N-1)p} & O_{(N-1)p \times p} \end{bmatrix} \tilde{D}_{s+1}. \tag{3.1.17}$$

Proof. (i): In Lemma 3.1.1 we have obtained a decomposition of the proper rational matrix $H_s(z)$ into its polynomial part, $\tilde{D}_s$, and its strictly proper rational part, $\tilde{H}_s(z)$. Then, taking into account that the matrix $H_s(z) = N_s(z)/d_s(z)$, and $d_s(z) = zd_0(z) = z^{r+1} + a_{r-1}z^r + \cdots + a_0 z$, we can consider the observable canonical invariant realization $(\tilde{C}_s, \tilde{A}_s, \tilde{B}_s)$ of $\tilde{H}_s(z)$. Hence we get the realization $(\tilde{C}_s, \tilde{A}_s, \tilde{B}_s, \tilde{D}_s)$ of $H_s(z)$, given in (3.1.12), (3.1.15), and (3.1.16).

(ii): The premultiplication of the matrix $\tilde{D}_{s+1}$ by

$$P_r = \begin{bmatrix} O_{p \times (N-1)p} & O_p \\ I_{(N-1)p} & O_{(N-1)p \times p} \end{bmatrix}$$

is equivalent to the following operations on the rows of $\tilde{D}_{s+1}$: the ith row, $i = 1, \dots, N-1$, takes the place of the $(i+1)$th row, and the Nth row is multiplied by zero and takes the place of the 1st row. Then

$$P_r \tilde{D}_{s+1} =$$

$$= \left[\begin{array}{cccc|cccc} O & O & \cdots & O & O & O & \cdots & O \\ O & O & \cdots & O & O & O & \cdots & O \\ O & O & \cdots & O & O & O & \cdots & [L_0^0]_{s+3,N-s-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & O & O & [L_0^0]_{N,2} & \cdots & [L_0^0]_{N,N-s-1} \\ \hline O & O & \cdots & O & [L_1^0]_{1,1} & [L_1^0]_{1,2} & \cdots & [L_1^0]_{1,N-s-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & [L_0^0]_{s,N} & [L_1^0]_{s,1} & [L_1^0]_{s,2} & \cdots & [L_1^0]_{s,N-s-1} \end{array}\right].$$

The postmultiplication of the matrix $\tilde{D}_s$ by

$$P_c = \begin{bmatrix} O_{(N-1)m \times m} & I_{(N-1)m} \\ O_m & O_{m \times (N-1)m} \end{bmatrix}$$

is equivalent to the following operations on the columns of $\tilde{D}_s$: the jth column, $j = 1, \dots, N-1$, takes the place of the $(j+1)$th column, and the Nth column is multiplied by zero and takes the place of the 1st column. Then

$$\tilde{D}_s P_c = \left[\begin{array}{cccc|cccc} O & O & \cdots & O & O & O & \cdots & O \\ O & O & \cdots & O & O & O & \cdots & O \\ O & O & \cdots & O & O & O & \cdots & [L_0^0]_{s+3,N-s-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & O & O & [L_0^0]_{N,2} & \cdots & [L_0^0]_{N,N-s-1} \\ \hline O & O & \cdots & O & [L_1^0]_{1,1} & [L_1^0]_{1,2} & \cdots & [L_1^0]_{1,N-s-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & [L_0^0]_{s,N} & [L_1^0]_{s,1} & [L_1^0]_{s,2} & \cdots & [L_1^0]_{s,N-s-1} \end{array}\right].$$

Hence $P_r \tilde{D}_{s+1} = \tilde{D}_s P_c$. ■

3.2. Periodic Realizations

Consider the Markov parameters $V_s(k) \in \mathbb{R}^{pN \times mN}$ of the invariant systems $(\tilde{C}_s, \tilde{A}_s, \tilde{B}_s, \tilde{D}_s)$, defined in (3.1.12)–(3.1.16):

$$V_s(k) = \begin{cases} \tilde{C}_s \tilde{A}_s^{k-1} \tilde{B}_s, & k \geqslant 1, \\ \tilde{D}_s, & k = 0. \end{cases} \tag{3.2.1}$$

We assume that $V_s(-1) = O$. These parameters are such that

$$H_s(z) = \tilde{C}_s \left(zI - \tilde{A}_s\right)^{-1} \tilde{B}_s + \tilde{D}_s = \sum_{k=1}^{\infty} V_s(k) z^{-k} + V_s(0),$$

$$s = 0, 1, \ldots, N-1. \tag{3.2.2}$$

Consider the N input-output invariant applications defined by the Markov sequences $\{V_s(k) : k \geqslant 0\}$, $s = 0, 1, \ldots, N-1$. By means of Definition 2.3 we shall construct the corresponding input-output N-periodic application provided that $V_s(k)$ satisfy the conditions of Lemma 2.1. This is done in the following result.

PROPOSITION 3.2.1. *The Markov parameters* $V_s(k)$ *given by* (3.2.1), (3.1.12)–(3.1.16) *satisfy the conditions in the expression* (2.16).

Proof. The observable canonical invariant realizations that we have constructed in Proposition 3.1.2, satisfy

$$V_s(k) = \tilde{C}_s \tilde{A}_s^{k-1} \tilde{B}_s = L_k^s, \qquad k \geqslant 1,$$

$$V_s(0) = \tilde{D}_s \tag{3.2.3}$$

for each $s = 0, 1, \ldots, N-1$. We split the proof of the conditions (2.16a–d) of Lemma 2.1 into two cases.

(I) Let us consider $k = 0$.

(2.16a): $1 \leqslant \alpha \leqslant N-1$, $2 \leqslant \beta \leqslant N$. By Part (ii) of Proposition 3.1.2 and our choice of $\tilde{D}_s$, given by (3.1.12), we observe that $[\tilde{D}_{s+1}]_{\alpha,\beta} = [\tilde{D}_s]_{\alpha+1,\beta-1}$, and hence $[V_{s+1}(0)]_{\alpha,\beta} = [V_s(0)]_{\alpha+1,\beta-1}$ holds.

(2.16b): $1 \leqslant \alpha \leqslant N-1$. Recall that $V_s(-1) = O$. Since the first column of blocks of $\tilde{D}_{s+1}$ is a zero block column, then $[V_{s+1}(0)]_{\alpha,1} = [V_s(-1)]_{\alpha+1,N}$.

(2.16c): $2 \leqslant \beta \leqslant N$. Using (3.2.3), we have $[V_{s+1}(0)]_{N,\beta} = [\tilde{D}_{s+1}]_{N,\beta}$ and $[V_s(1)]_{1,\beta-1} = [L_1^s]_{1,\beta-1}$. From (3.1.11) and (3.1.12) we can see that the last row of blocks of $\tilde{D}_{s+1}$ is given by $[O, [L_0^0]_{s+1,\ N-s+1}, \dots, [L_0^0]_{s+1,N}, [L_1^0]_{s+1,1}, \dots, [L_1^0]_{s+1,N-s-2}, [L_1^0]_{s+1,N-s-1}]$ and the first row of blocks of the matrix L_1^s is given by $[[L_0^0]_{s+1,N-s+1}, \dots, [L_0^0]_{s+1,N}, [L_1^0]_{s+1,1}, \dots, [L_1^0]_{s+1,N-s-1}, [L_1^0]_{s+1,N-s}]$. Then we obtain that $[V_{s+1}(0)]_{N,\beta} = [V_s(1)]_{1,\beta-1}$.

(2.16d): Using (3.2.3) and (3.1.12), we conclude that $[V_{s+1}(0)]_{N,1} = O = [V_s(0)]_{1,N}$.

(II) Now consider $k \geqslant 1$.

(2.16a): $1 \leqslant \alpha \leqslant N-1$, $2 \leqslant \beta \leqslant N$. According to (3.1.11) we have $[L_k^{s+1}])_{\alpha,\beta} = [L_k^s]_{\alpha+1,\beta-1}$ and then $[V_{s+1}(k)]_{\alpha,\beta} = [V_s(k)]_{\alpha+1,\beta-1}$.

(2.16b): $1 \leqslant \alpha \leqslant N-1$. If $k=1$ then $V_s(k-1) = V_s(0) = \tilde{D}_s$ and $V_{s+1}(k) = V_{s+1}(1) = L_1^{s+1}$. From (3.1.12) we have that

$$\operatorname{col}\left[O, \left[L_0^0\right]_{s+2,\,N-s}, \dots, \left[L_0^0\right]_{N,\,N-s}, \left[L_1^0\right]_{1,\,N-s}, \dots, \left[L_1^0\right]_{s,\,N-s}\right] \quad (3.2.4)$$

is the last column of blocks of $V_s(0)$, and from (3.1.11)

$$\operatorname{col}\left[\left[L_0^0\right]_{s+2,\,N-s}, \dots, \left[L_0^0\right]_{N,\,N-s}, \left[L_1^0\right]_{1,\,N-s}, \dots, \left[L_1^0\right]_{s+1,\,N-s}\right] \quad (3.2.5)$$

is the first column of blocks of $V_{s+1}(1)$. Then we obtain that $[V_{s+1}(1)]_{\alpha,1} = [V_s(0)]_{\alpha+1,N}$. If $k>1$ then $V_s(k-1) = L_{k-1}^s$ and $V_{s+1}(k) = L_k^{s+1}$. From (3.1.11) we obtain that

$$\operatorname{col}\left[\left[L_{k-1}^0\right]_{s+1,\,N-s}, \dots, \left[L_{k-1}^0\right]_{N,\,N-s}, \left[L_k^0\right]_{1,\,N-s}, \dots, \left[L_k^0\right]_{s,\,N-s}\right] \quad (3.2.6a)$$

is the last column of blocks of $V_s(k-1)$ and

$$\operatorname{col}\left[\left[L_{k-1}^0\right]_{s+2,\,N-s}, \dots, \left[L_{k-1}^0\right]_{N,\,N-s}, \left[L_k^0\right]_{1,\,N-s}, \dots, \left[L_k^0\right]_{s+1,\,N-s}\right] \quad (3.2.6b)$$

is the first column of blocks of $V_{s+1}(k)$. Then $[V_{s+1}(k)]_{\alpha,1} = [V_s(k-1)]_{\alpha+1,N}$.

(2.16c): $2 \leqslant \beta \leqslant N$. Observe that the $N-1$ last blocks of the last block row of the matrix L_k^{s+1} are equal to the $N-1$ first blocks of the first row of

blocks of the matrix L^s_{k+1}. Then $[V_{s+1}(k)]_{N,\beta} = [V_s(k+1)]_{1,\beta-1}$.
(2.16d): We have that $[V_{s+1}(k)]_{N,1} = [L^{s+1}_k]_{N,1} = [L^0_k]_{s+1,N-s}$, and since $[V_s(k)]_{1,N} = [L^s_k]_{1,N} = [L^0_k]_{s+1,N-s}$, the equality holds.

From the above proposition, the invariant applications $\{V_s(k): k \geqslant 0\}$ satisfy the condition (2.16). Then the N-periodic Markov sequences constructed by using Definition 2.3 have the translation and periodicity properties. Therefore, by Definition 2.1(a) we can consider the input-output periodic application.

$$y(k+s) = \sum_{j=0}^{k-1} V_s(k, k-j)u(j+s), \qquad k \geqslant 1, \quad s \in \mathbb{Z}. \quad (3.2.7)$$

From (3.2.1), $V_s(0) = \tilde{D}_s$, and by (3.1.12), $[\tilde{D}_s]_{\alpha,\beta} = O$ for $\alpha + \beta < N + 2$. Then we can use Corollary 2.1 and deduce that the input-output invariant applications associated with (3.2.7) are given by the Markov parameters $\{V_s(k): k \geqslant 0\}$ in the expression (3.2.1).

PROPOSITION 3.2.2. *Let $m(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_0$ be the minimal polynomial of the matrix $\tilde{A}_0$. Then the Markov parameters $V_s(k)$ given by* (3.2.1) *satisfy the recurrence equation*

$$V_s(k+r+1) + a_{r-1}V_s(k+r) + \cdots + a_0 V_s(k+1) = O, \qquad k \geqslant 1,$$

for each $s = 0, 1, \ldots, N-1$.

Proof. From (3.1.11) we have that $\overline{m}(\lambda) = \lambda^{r+1} + a_{r-1}\lambda^r + \cdots + a_0\lambda$ is the minimal polynomial of $\tilde{A}_s$, $s = 1, 2, \ldots, N-1$. Then $\tilde{A}_s^{r+1} + a_{r-1}\tilde{A}_s^r + \cdots + a_0\tilde{A}_s = O$, $s = 0, 1, \ldots, N-1$. Premultiplying by $\tilde{C}_s\tilde{A}_s^{k-1}$ and postmultiplying by $\tilde{B}_s$, we obtain $\tilde{C}_s\tilde{A}_s^{k+r}\tilde{B}_s + a_{r-1}\tilde{C}_s\tilde{A}_s^{k+r-1}\tilde{B}_s + \cdots + a_0\tilde{C}_s\tilde{A}_s^k\tilde{B}_s = O$, $k \geqslant 1$, and the recurrence relation holds. ■

Since the N input-output invariant applications defined by the Markov sequences given in (3.2.1) satisfy the above recurrence equation, then the input-output periodic application (3.2.7) satisfies the same recurrence equation (see Proposition 2.2). So, from Theorem 2.1, there exists a periodic realization of (3.2.7). Denote this periodic realization by $(G(\cdot), E(\cdot), F(\cdot))_N$ and let (G_s, E_s, F_s, H_s), $s \in \mathbb{Z}$, be the associated invariant systems. From Proposition 2.1 we obtain that

$$V_s(k) = \begin{cases} G_s E_s^{k-1} F_s, & k \geqslant 1, \\ H_s, & k = 0. \end{cases}$$

Now, using the relation (3.2.2) we obtain

$$H_s(z) = \sum_{k=1}^{\infty} V_s(k) z^{-k} + V_s(0) = G_s(zI - E_s)^{-1} F_s + H_s.$$

Hence, by Definition 2.4, $(G(\cdot), E(\cdot), F(\cdot))_N$ is a periodic realization of the collection of N-periodic rational matrices given in (3.1.1).

The next theorem summarizes all the above-obtained results.

THEOREM 3.2.1. *Let* $\{H_s(z),\ s \in \mathbb{Z}\}$, $H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN \times mN}(z)$, $N \in \mathbb{Z}^+$, *be a sequence of periodic rational matrices. Then there exists an* N*-periodic system* $(\tilde{C}(\cdot), \tilde{A}(\cdot), \tilde{B}(\cdot))_N$ *which is a periodic realization of* $\{H_s(z),\ s \in \mathbb{Z}\}$ *if and only if* $H_{s+1}(z) = S(z) H_s(z) T(z)$, $s \in \mathbb{Z}$, *where*

$$S(z) = \begin{bmatrix} O & I_{(N-1)p} \\ zI_p & O \end{bmatrix}, \qquad T(z) = \begin{bmatrix} O & I_{(N-1)m} \\ z^{-1} I_m & O \end{bmatrix},$$

and $H_0(z)$ *is proper with strictly lower block-triangular polynomial part with respect to the antidiagonal.*

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